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ABSTRACT

Tandem-mirror trapped-particle modes are studied in a model system consisting of two connected square wells representing the solenoid and the end cells. Collisions are described by a Lorentz operator. A dispersion relation that is valid for arbitrary ν/ω (ω = wave frequency, ν = collision frequency) is derived. Four limits are investigated. When $\varepsilon \equiv \nu R_{am}/\omega \ll 1 + p_i/p_e$, where R_{am} is the mirror ratio separating electrons trapped in the anchor from those passing to the solenoid and p_e and p_i are the fractions of passing electrons and ions, then collisions destabilize a trapped particle mode that is stable in the collisionless limit; the growth rate is proportional to $\varepsilon^{1/2}$ for $\varepsilon \ll 1$ and $\varepsilon/\ln \varepsilon$ for $1 + p_i/p_e \gg \varepsilon \gg 1$. When $\varepsilon \gg 1 + p_i/p_e$, the trapped particle mode becomes a weakly growing drift wave with growth rate proportional to $\varepsilon^{-1}\ln \varepsilon$ for $\nu/\omega \ll 1$ and ν^{-1} for $\nu/\omega \gg 1$; additionally identified are two flute modes, one of which is unstable for some parameters, and a strongly damped high-frequency mode.

I. INTRODUCTION

The requirement that tandem mirrors be stable to collisionless trapped particle modes[1] sets a lower bound on the number of passing particles needed to connect regions of good and bad curvature. This requirement significantly impacts the choice of operating scenario and plasma parameters for tandem mirrors with axisymmetric throttles, such as TARA, MFTF-B, and the MARS reactor design. In present-day experiments (TMX-U, TARA, Phaedrus, GAMA-10) the electron collision frequency is comparable with predicted trapped-particle mode frequencies; thus collisional effects are strong in such machines and must be understood in order to diagnose the trapped particle mode.

Collisions have been considered previously by Lane[2], who used a Lorentz (pitch-angle scattering) collision operator to derive results valid for very weak ($\varepsilon \ll 1$) and strong ($\nu/\omega \gg 1$) collisionality in the absence of any axial variation of the equilibrium electrostatic potential Φ. The present paper is the second in a series of three intended to more thoroughly treat collisions. In the first[3], a dispersion relation was derived using a general (unspecified) collision operator, and retaining variation in Φ, in the limit of large collisionality. The present paper restricts attention to Lorentz collisions and treats the magnetic field as coupled square wells, but derives a dispersion relation for arbitrary collisionality and arbitrary ratio of well lengths, again retaining effects of variation in Φ. Because of the neglect of energy scatter, the results are of physical significance only at low collisionality or when there is no significant variation in Φ. The third paper[4] addresses this problem by considering the full collision operator, but attention is restricted to the case where the good curvature (anchor) region, or more precisely the region in which the fluctuating potential varies from its central-cell value, is short compared to the overall machine length. In all three papers, the effect of Φ variations on velocity-space boundaries and the transformation of velocities from one axial region to another are retained. Effects arising through the axial variation of drift frequencies are neglected in this paper but are retained in Ref. 3. The latter can have an important effect on these modes, and will be included in the general collisionality analysis in the future.

The present paper is organinzed as follows. Section II presents the physical model and the kinetic equation governing trapped and passing groups of a collisional species (nominally, electrons). In Section III, the perturbed distribution function and perturbed density are determined for arbitrary collisionality, in terms of Legendre functions. A formal expression for the dispersion relation at arbitrary collisionality is given. The Legendre functions reduce to elementary functions in three collisionality limits, and these are discussed in Sections IV-VI. In the low collisionality regime ($\varepsilon \ll 1$), collisions produce small shifts in the collisionless mode frequencies. In particular, collisions destabilize one of two collisionless modes in the case where there are sufficient passing particles to render the collisionless modes stable; the growth rate varies as $\varepsilon^{1/2}$. This is developed in Sec. IV. In Sec. V, the intermediate regime ($R_{am}^{-1} \ll \nu/\omega \ll 1$), the Legendre functions reduce to the small-argument limits of Bessel functions. Two sub-orderings are obtained in which the collisional perturbation to the fluctuating charge density is small or large compared to the collisionless value according to whether $\varepsilon/\ell n\ \varepsilon$ is smaller or larger than $1 + P_i/P_e$. In the former case collisions are again a perturbation on the collisionless results, but the residual growth rate for the stable collisionless trapped particle mode is now proportional to $\varepsilon/\ell n\ \varepsilon$. In the latter case the modes are significantly altered by collisions. We identify four roots to the dispersion relation: a weakly unstable electron drift wave with growth rate proportional to $\varepsilon^{-1}\ \ell n\ \varepsilon$, two flute modes of which, for some parameters, one may be unstable (with growth rate proportional to $\varepsilon^{-1}\ \ell n\ \varepsilon$), and a high-frequency damped mode. The latter set of modes persists into the high collisionality regime ($\nu/\omega >> 1$, Section VI), except that the scaling of the growth rates no longer involves the logarithm. A discussion of the results is given in Section VII.

II. THE MODEL

In this paper we wish to allow for an arbitrary ratio of the wave frequency, ω, to the electron collision frequency, ν, while retaining only those elements of the tandem mirror geometry that are essential to the study of trapped particle modes. Hence, we model the tandem mirror as a sequence of connected square wells. The center cell has a length L_c, magnetic field B_c, and an average curvature described by the curvature drift frequency ω_{Dc}. At each end of the center cell there is an anchor cell of length $L_a/2$, magnetic field B_a, and an averaged curvature drift frequency ω_{Da}. The equilibrium electrostatic potential in the center cell is taken to be zero, while the potential in the anchor cell is Φ_a. The mirror ratio of the magnetic mirror separating the center and anchor cells is R_{cm}, while the potential barrier separating these cells is Φ_{cm}. The plasma densities in the center and anchor cells are n_c and n_a respectively.

Electron collisions affect trapped particle modes because they allow scattering between the trapped and passing regions of phase space. In the large mirror ratio limit, $R_{cm}^{1/2} \gg |e\Phi_{cm}/T|$, it is reasonable to ignore energy diffusion and drag, and model collisions with a Lorentz collision operator. This large mirror ratio limit is appropriate in analyzing the throttle-coil operating senarios of TMX-U[5] and MFTF-B[6], where trapped particle modes are potentially most dangerious due to the small fraction of passing particles. Using this approximation, the pitch-angle dependence of the non-adiabatic perturbation in the distribution function of electrons trapped in region j (j=c,a), h_j, is described by a bounce-averaged kinetic equation,

$$i\nu_j \frac{\partial}{\partial\lambda_j} (1-\lambda_j^2) \frac{\partial}{\partial\lambda_j} h_j -(\omega-\omega_{Dj})h_j = (\omega-\omega_{*e})\phi_j F_m \ , \qquad (1)$$

where $\phi_j \simeq -|e|\tilde{\phi}_j/T_e$, $\tilde{\phi}_j$, ν_j, and λ_j are the perturbed potential, collision frequency, and cosine of the pitch angle in cell j; $\omega_{*e} \equiv \Omega_{*0}[1+\eta_e(m_e v^2/2T_e - 3/2)]$ is the diamagnetic drift frequency, $\Omega_{*0} \equiv -(\ell c T_e)/(|e|n)\partial n/\partial\Psi$, ℓ is the azimuthal mode number, T_e is the electron temperature (which is assumed to be the same function of Ψ in the center and anchor cells), $\eta_e \equiv (n/T_e)(\partial T_e/\partial\Psi)/(\partial n/\partial\Psi)$, and F_m is the zero order distribution function of the electrons, which is assumed to be locally Maxwellian.

The passing electrons suffer collisions in both the center cell and the anchor cell. Hence, the bounce averaged kinetic equation for the passing electrons is

$$\overline{C}(h_p) - (\omega-\overline{\omega}_D)h_p = (\omega-\omega_{*e})\, \overline{\phi}\, F_M, \tag{2}$$

where barred quantities are bounce averaged over the passing particle orbits,

$$\overline{\omega}_D = \frac{1}{\tau_b} (\tau_c\omega_{Dc} + \tau_a\omega_{Da}) \; , \tag{3}$$

$$\overline{\phi} = \frac{1}{\tau_b} (\tau_c\phi_c + \tau_a\phi_a), \tag{4}$$

and

$$\overline{C}(h_p) = \frac{1}{\tau_b} [\tau_c C_c(h_p) + \tau_a C_a(h_p)] \; . \tag{5}$$

τ_b is the time to traverse the tandem mirror, $\tau_b = \tau_c + \tau_a$, and τ_c and τ_a are the times to traverse the center cell and both anchor cells, respectively, $\tau_j = L_j/v_j\lambda_j$. $C_j(h)$ is the first term on the left hand side of Eq. (1).

We choose to describe the passing particle distribution in center cell co-ordinates. We note that

$$\left(1-\lambda_a{}^2\right) = \sigma\left(1-\lambda_c{}^2\right) , \tag{6}$$

where $\varepsilon \equiv 1/2mv_c{}^2$ and $\sigma = \varepsilon R_{ca}/(\varepsilon - q\Phi_a)$ with $R_{ca} = B_a/B_c$; and thus that $C_a(h_p)$ can be written as

$$C_a(h_p) = i\nu_a \frac{\lambda_a}{\sigma\lambda_c} \frac{\partial}{\partial\lambda_c} \frac{\lambda_a}{\lambda_c} \left(1-\lambda_c{}^2\right) \frac{\partial h_p}{\partial\lambda_c} \tag{7}$$

where $\lambda_a(\lambda_c)$ is given by Eq. (6). The averaged operator $\overline{C}$ takes on a particularly simple form (Legendre operator) in either of two limits: (1) $R_{ca} = 1$, $\Phi_a = 0$, in which case $\lambda_a = \lambda_c$ for arbitrary R_{cm}, and (2) $R_{cm} >> 1$ so that, for passing particles, $\lambda_a, \lambda_c = 1 + O\left(R_{cm}{}^{-1}\right)$. In either case, we find

$$\overline{C}(h_p) \simeq i\overline{\nu} \frac{\partial}{\partial\lambda_c} \left(1-\lambda_c{}^2\right) \frac{\partial}{\partial\lambda_c} h_p , \tag{8}$$

where

$$\overline{\nu} = (\nu_c\tau_c + \nu_a\tau_a/\sigma)/\tau_b \tag{9}$$

and henceforth $\tau_j \equiv L_j/v_j$.

III. PERTURBED RESPONSE AND DISPERSION RELATION FOR ARBITRARY ν/ω

Comparing Eqs. (1), (2), and (8) we see that, in the large mirror ratio limit, each group of particles in the tandem mirror obeys an equation of the form of Eq. (1) where j can now take on the value p (for passing paricles). The bounce averaged drift frequency, potential, and collision frequency for the passing particles are given by Eqs. (3), (4), and (9). If we neglect the λ dependence of ω_{Dj}, Eq. (1) is an inhomogeneous Legendre's equation of order zero. The general solution to this equation is the sum of a particular solution to the inhomogeneous equation together with a solution to the homogeneous equation chosen to satisify the boundary conditions in each region. A particular solution to the inhomogeneous equation that is independent of λ is

$$h_j^0 = -\frac{\omega-\omega_{*e}}{\omega-\omega_{Dj}}\,\phi_j F_M \ . \tag{10}$$

The solutions to the homogeneous equation are the Legendre functions $P^0_{\alpha_j}(\lambda_j)$ and $Q^0_{\alpha_j}(\lambda_j)$, where α_j is a root of the quadratic equation

$$\alpha_j(\alpha_j+1) = i\,\frac{\omega-\omega_{Dj}}{\nu_j} \ . \tag{11}$$

The boundary condition on the trapped particles is that $dh/d\lambda_j=0$ at $\lambda_j=0$, while for passing particles $h_p(\lambda)$ must be non-singular at $\lambda=1$. Hence, a homogeneous solution for particles trapped in the center or anchor cells that satisfies the boundary condition at $\lambda=0$ is

$$\tilde{h}_j(\lambda_j) = Q^0_{\alpha_j}(\lambda_j) - \frac{\pi}{2}\cot\left(\frac{\pi}{2}\alpha_j\right)P^0_{\alpha_j}(\lambda_j) \quad , \tag{12}$$

while the homogeneous solution in the passing particle region of phase space is

$$\tilde{h}_p(\lambda_c) = P^0_{\alpha_p}(\lambda_c) \ . \tag{13}$$

It is helpful to introduce a normalized solution, $\hat{h}$, given by

$$\hat{h}_j(\lambda_j) \equiv \frac{\tilde{h}_j\ (\lambda_j)}{\tilde{h}_j\ (\lambda_{js})} \ , \tag{14}$$

where λ_{js} is the value that λ_j takes on the separatrix in region j. We adopt the convention that $\lambda_{ps} = \lambda_{cs}$. The non-adiabatic part of the perturbed distribution function in region j may then be written as

$$h_j(\lambda_j) = A_j\ \hat{h}_j(\lambda_j) + h^0_j \ . \tag{15}$$

It remains to evaluate the constants A_c, A_a, and A_p. This is accomplished by requiring the perturbed distribution function to be continuous across the separatrix, which yields

$$A_p + h^0_p = A_c + h^0_c \ , \tag{16}$$

and

$$A_p + h^0_p = A_a + h^0_a \ ; \tag{17}$$

and demanding that there be no net flux of particles into the separatrix, which implies

$$\nu_c \tau_c \frac{\partial h_c}{\partial \lambda_c} + \nu_a \tau_a \frac{\partial h_a}{\partial \lambda_a} - \bar{\nu} \tau_b \frac{\partial h_p}{\partial \lambda_c} = 0 \ . \tag{18}$$

Equations (16) thru (18) yield

$$A_p = \frac{\hat{h}_c'(h_p^0 - h_c^0) + \sigma\ \delta\ \hat{h}_a'(h_p^0 - h_a^0)}{(1+\delta)\hat{h}_p' - \hat{h}_c' - \sigma\ \delta\ \hat{h}_a'} \ , \tag{19}$$

$$A_c = A_p + h_p^0 - h_c^0 \ , \tag{20}$$

and

$$A_a = A_p + h_p^0 - h_a^0 \ ; \tag{21}$$

where

$$\hat{h}_j' \equiv \frac{\partial \hat{h}_j}{\partial \lambda_{js}} \ , \tag{22}$$

and

$$\delta \equiv \frac{L_a n_a \sigma}{L_c n_c R_{ca}^2} \ . \tag{23}$$

Note that the particular solution, h_j^0, is also the solution to the collisionless problem. Hence, we may write the perturbed electron density in the j^{th} cell as a sum of the collisonless responce, $\tilde{n}_{ej}^0$, and a collisional responce,

$$\tilde{n}_{ej}^{\nu} \cong 4\pi \int v_j^2 dv_j \, A_j \int_0^{\lambda_{js}} d\lambda_j \, \hat{h}_j(\lambda_j) + 4\pi \int v_c^2 dv_c \, (\sigma_j R_{cj})^{1/2} A_p \int_{\lambda_{cs}}^{1} d\lambda_c \, \hat{h}_p(\lambda_c) \; , \tag{24}$$

where $R_{cj} \equiv B_j/B_c$, $\sigma_a \equiv \sigma$, and $\sigma_c \equiv 1$. In deriving (24) we have set $\lambda_a/\lambda_c = 1$ consistent with the approximations made in deriving Eq. (8). In some applications (e.g., when the collision frequency is treated as a small parameter) it is convenient make the replacement

$$\hat{h}_j = i \frac{\nu_j}{\omega-\omega_{Dj}} \frac{\partial}{\partial \lambda_j} (1-\lambda_j^2) \frac{\partial}{\partial \lambda_j} \hat{h}_j \; , \tag{25a}$$

$$= -i \frac{\nu_j}{\omega-\omega_{Dj}} \frac{\partial}{\partial \lambda_j} \hat{h}_j^1 \tag{25b}$$

where $\hat{h}_j^1 = (1-\lambda_j^2)^{1/2} \, \tilde{h}_j^1(\lambda_j)/\tilde{h}_j(\lambda_{js})$ and

$$\tilde{h}_j^1(\lambda_j) = Q_{\alpha_j}^1(\lambda_j) - \frac{\pi}{2}\cot\left(\frac{\pi}{2}\alpha_j\right) P_{\alpha_j}^1(\lambda_j) \quad , \quad j=c,a \tag{26}$$

$$\tilde{h}_p^1(\lambda_c) = P_{\alpha_p}^1(\lambda_c) \; . \tag{27}$$

An integral over λ then yields

$$\tilde{n}_{ej}^{\nu} = -4\pi i \int v_j^2 dv_j \, A_j \frac{\nu_j}{\omega-\omega_{Dj}} \hat{h}_j^1(\lambda_{js})$$

$$+ 4\pi i \int v_c{}^2 dv_c \, (\sigma_j R_{cj})^{1/2} A_p \frac{\nu_p}{\omega-\omega_{Dp}} \hat{h}_p^1(\lambda_{cs}) \quad . \tag{28}$$

In deriving (25b) we have used the identity $L_j^1(\lambda) = -(1-\lambda^2)^{1/2}\, \partial L_j^0/\partial\lambda$ for Legendre functions L = P or Q.

Observing from Eqs. (10), (19) and (24) or (28) that $\tilde{n}^{\nu}_{ej}$ is a linear function of ϕ_c and ϕ_a, we can define a collisional susceptibility tensor $\overleftrightarrow{\chi}^{\nu}$ by the relation $(1/bN)\tilde{N}_j^{\nu} = \sum \chi^{\nu}_{jk}\phi_k$ where $\tilde{N}_j^{\nu} = \tilde{n}_j^{\nu} L_j/B_j$ is the collisional perturbation in the number of particles per unit flux in region j, $N = \int n\, ds/B$ is the number of particles per unit flux in both the center and anchor cells, and $b \equiv k_{\perp c}^2 T_{\perp p}/m\omega_{cic}^2$; here ω_{cic} is the ion cyclotron frequency in the central cell. We can now formally write the dispersion relation for arbitrary collisionality; it is

$$|\overleftrightarrow{\chi}^0 + \overleftrightarrow{\chi}^{\nu}| = 0 \tag{29}$$

where $\overleftrightarrow{\chi}^0$ is the collisionless susceptibility; it is given explicitly in Eq. (41).

IV. LOW COLLISIONALITY REGIME

The low collision-frequency limit ($\nu_c/\omega \ll R^{-1} < 1$)) is obtained from the large-α, finite-θ asymptotic expansions for the Legendre functions:

$$P_\alpha^\mu(\cos\,\theta) = \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+3/2)}\left(\frac{\pi}{2}\sin\,\theta\right)^{-1/2}\cos\,\Psi + O(\alpha^{-1}) \tag{31a}$$

$$\cong \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+3/2)}\,(2\pi\;\sin\,\theta)^{-1/2}\,e^{-i\Psi} \tag{31b}$$

$$Q_\alpha^\mu(\cos\,\theta) - \frac{\pi}{2}\cot\left(\frac{\pi\alpha}{2}\right)P_\alpha^\mu \cong \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+3/2)}\left(\frac{\pi}{2\,\sin\,\theta}\right)^{-1/2} e^{-i\Psi} \tag{32}$$

where

$$\Psi \equiv \left(\alpha+\frac{1}{2}\right)\theta - \frac{\pi}{4} + \frac{\mu\pi}{2}$$

Eqs. (31b) and (32) are derived assuming that Im $\alpha \to +\infty$ as $\alpha \to \infty$. Since for our application $\alpha_j \cong [i(\omega-\omega_{Dj})/\nu_j]^{1/2}$, Eqs. (31b) and (32) are valid provided that the appropriate square root is taken. Also, validity requires $\theta >> \alpha^{-1}$; thus, at $\theta = \theta_{js} \sim R_{jm}^{-1}$, the requirement is $\nu_c/\omega < R_{jm}{}^{-1/2}$, which defines the extent of the low-collisionality regime. From (31) and (32) it follows that

$$h_p = -i\;\alpha_p\;\sin\,\theta_c \tag{34}$$

$$h_j = i\;\alpha_j\;\sin\,\theta_j \quad ,\; j=c,a \tag{35}$$

Then from Eqs. (28) and (19)-(21), we find, to leading order in ω_D/ω,

$$\frac{1}{bN}\begin{pmatrix} \tilde{N}_{ec}^\nu \\ \tilde{N}_{ea}^\nu \end{pmatrix} = \begin{pmatrix} \chi^\nu & -\chi^\nu \\ -\chi^\nu & \chi^\nu \end{pmatrix}\begin{pmatrix} \phi_c \\ \phi_a \end{pmatrix} \tag{36}$$

or $\tilde{\underset{\sim}{N}}_e^\nu/(bN) = \overset{\leftrightarrow}{\chi}{}^\nu \underset{\sim}{\phi}$, where

$$\chi^{\nu} = \exp(i\pi/4)\ P_{\phi}(N_a/N)\ \langle(1-\omega_*/\omega)\ \ell_c\ (\nu_a/\omega)^{1/2}\ K \sin\theta_{as}\rangle_{\phi} \quad (37)$$

with $\ell_r \equiv \tau_j/\tau$,

$$P_{\phi} = \frac{\int_{v_{\phi}}^{\infty} v_a^2\ F\ dv_a}{b\int_0^{\infty} v_a^2\ F\ dv} = b^{-1}\left[2\left(\frac{e\Phi_{am}}{\pi T_e}\right)^{1/2} \exp(-e\Phi_{am}/T_e) + \mathrm{erfc}\left((e\Phi_{am}/T_e)^{1/2}\right)\right] \quad (38)$$

and

$$K = \frac{\ell_c\left(\frac{\nu_p}{\nu_c}\right)^{1/2} + 1 + \ell_a\left(\frac{\sigma\nu_p}{\nu_a}\right)^{1/2}}{\left(\frac{\nu_p}{\nu_c}\right)^{1/2} + \ell_c + \ell_a\left(\frac{\nu_a}{\sigma\nu_c}\right)^{1/2}} \quad (39)$$

$$\langle x\rangle_{\phi} \equiv \frac{\int_{v_{\phi}}^{\infty} v_a^{\,2}\ dv_a\ Fx}{\int_{v_{\phi}}^{\infty} v_a^{\,2}\ dv_a\ F} \quad (40)$$

where $v_{\phi} = (2e\Phi_{am}/T_e)^{1/2}$. In evaluating $\omega^{1/2}$, the branch cut is to be taken on the negative imaginary axis.

The dispersion relation and mode structure are given by quasineutrality, i.e., $\tilde{\underset{\sim}{N}}_i^0 - \tilde{\underset{\sim}{N}}_e^0 - \tilde{\underset{\sim}{N}}_e^{\nu} = 0$, or $(\ddot{\chi}^0 + \ddot{\chi}^{\nu})\ \underset{\sim}{\phi} = 0$, where, from Eq. (10), it follows that, to leading order in ω_{Dj}/ω, the fraction of passing particles, and $k^2\ \rho_i^{\,2}$, $\ddot{\chi}^0$ has the form

$$\overleftrightarrow{\chi}^0 = \begin{pmatrix} \chi_{cc} + \chi_{ca} & -\chi_{ca} \\ -\chi_{ca} & \chi_{aa} + \chi_{ca} \end{pmatrix}$$

$$\chi_{ca} = \sum_j P_j \left(1 - \frac{\Omega_{*jp}}{\omega}\right)$$

$$\chi_{cc} = \frac{N_c}{N}\left(1 - \frac{\Omega_{*ic}}{\omega} + \frac{\gamma_c^2}{\omega^2}\right) \tag{41}$$

$$\chi_{aa} = \frac{N_a}{N}\left(1 - \frac{\Omega_{*ia}}{\omega} - \frac{\omega_a^2}{\omega^2}\right)$$

with $P_j = (N_{sj}T_e/NbT_{jp})\langle\tau_a\tau_c/\tau^2\rangle_{pj}$, N_{jp} is the number of particles of species j on a flux tube, $\langle\ \rangle_{pj}$ is the phase-space average over passing particles:

$$\langle a\rangle_{jp} \cong \frac{\int_p d^3v_c \left(L_c a_c + L_p a_p (\sigma/R_{ca})^{1/2}\right) F_j}{\int_p d^3v_c \left(L_c + L_p(\sigma/R_{ca})^{1/2}\right) F_j},$$

the averaged diamagnetic drift frequencies are defined by $\Omega_{*jr} = -(\ell c/qn_r T_{jr})\ \partial n_r/\partial\Psi$ for $r = c,a$ and

$$\Omega_{*jp} = -\frac{\ell c T_{jp}}{q\hat{n}_{jp}}\frac{\partial\hat{n}_{jp}}{\partial\Psi}\frac{\langle\zeta_{jp}\tau_a\tau_c/\tau^2\rangle_{jp}}{\langle\tau_a\tau_c/\tau^2\rangle_{jp}},$$

$$\zeta_{jr} = 1 + \eta_{jr}\left(\frac{mv_r^2}{2T_{jr}} - \frac{3}{2}\right)$$

$\eta_{jr} = d\ \ln T_{jr}/d\ \ln n_{jr}$,

ℓ is the azimuthal mode number, p_{jr} is the pressure of species j in cell r, Ψ is the magnetic flux, $\hat{n}_{jp}$ is the full density associated with the Maxwellian distribution function describing the passing particles of species j, $-\gamma_c^2$ and ω_a^2 are the squares of the MHD frequencies in the cntral cell and anchor,

$$b\gamma_c{}^2 \equiv \sum_j \Omega_{*jc}\langle\zeta_{jc}\omega_{Djc}\rangle_{jc}\,\frac{T_{jc}}{T_e}$$

$$b\omega_a{}^2 \equiv \sum_j \Omega_{*ja}\langle\zeta_{ja}\omega_{Dja}\rangle_{ja}\,\frac{T_{ja}}{T_e} \quad ,$$

sums on j run over electron and ion species, and $\langle\ \rangle_{jr}$ denotes the F-weighted average over the trapped portion of velocity space for species j in cell r (r = c,a). Note that the ζ's can be replaced by one if $\Phi_{ca} = \Phi_{cm} = 0$ and $R_{ca} = 1$; otherwise, there is extra charge uncovering associated with the temperature gradient. It is assumed that the ion distribution functions in all regions of phase space (trapped in solenoid, trapped in anchor, passing) are Maxwellian, but generally with different densities and temperatures.

The dispersion relation is given by $|\overleftrightarrow{\chi}^0 + \overleftrightarrow{\chi}^\nu| = 0$; since $|\overleftrightarrow{\chi}^\nu| \equiv 0$, this can be written in the form

$$D^0 + D^\nu = 0 \qquad (42)$$

where

$$D^0 = \chi_{aa}\chi_{cc} + \chi_{ca}(\chi_{cc} + \chi_{aa}) \tag{43}$$

and

$$D^\nu = \chi^\nu(\chi_{cc} + \chi_{aa}) \tag{44}$$

Estimating $\hat{\Omega}_{*jr} \sim \Omega_{*0}$, we see that the components of $\overleftrightarrow{\chi}^\nu$ are smaller than those of $\overleftrightarrow{\chi}^0$ and thus D^ν is smaller than D^0 by order $\varepsilon^{1/2}\, P_e/(P_e + P_i + 1)$, where $\varepsilon \equiv (\nu_a\; R_{am}/\Omega_*)$. Thus D^ν can be treated as a perturbation. Then the shift in frequency $\delta\omega$ from the collisionless result ω_0 (where ω_0 satisfies $D^0 = 0$) is given by

$$\delta\omega \cong \frac{-D^\nu}{\frac{d}{d\omega} D^0} \times [1 + O(\varepsilon)] \tag{45}$$

From the mode equation $(\overleftrightarrow{\chi}^0 + \overleftrightarrow{\chi}^\nu)\, \underset{\sim}{\phi} = 0$, the dispersion relation (42), and the form of $\overleftrightarrow{\chi}^0$, $\overleftrightarrow{\chi}^\nu$ it follows that

$$\frac{\phi_a}{\phi_c} = -\frac{\chi_{cc}}{\chi_{aa}} \tag{46}$$

In the collisonless limit, the dispersion relation is usually obtained[1] assuming that $\phi_a = 0$ and thus solving the condition $\chi_{cc} + \chi_{ca} = 0$ for quasineutrality in the central cell. Inspection of Eqs. (41) and (46) shows that this assumption is justified if $\omega_a{}^2/\omega^2$ is much greater than 1 and the P_j. This limit obtains in a machine which is strongly MHD-stable, $\omega_a^2 \gg \ell_c\gamma_c^2/\ell_a$. In this limit, $\omega_0 \cong (2A)^{-1}\,[-\Omega_{*0}B$

$\pm\ (\Omega_{*0}^2B^2-4A\gamma_c^2)^{1/2}]$, with $A = (N_c/N) + \Sigma P_{jp}$, $\Omega_{*0}B = -(N_c/N)\Omega_{*1c} + \Sigma P_j\Omega_{*jp}$, and

$$\delta\omega \cong -\frac{\chi_{cc}^{\nu}}{\frac{d}{d\omega}(\chi_{cc}+\chi_{ca})} \tag{47a}$$

$$= -\frac{\omega_0^{1/2}\exp(i\pi/4)\ P_\phi(N_c/N)\langle\ell_a\left(1-\frac{\omega_*}{\omega_0}\right)v_a^{1/2}K\ \sin\theta_{as}\rangle_c}{2A+\frac{B\Omega_{*0}}{\omega_0}} \tag{47b}$$

If the collisionless mode is stabilized by charge uncovering or finite gyroradius effects so that $\Omega_{*0}^2B^2 > 4A\gamma_c^2$, and $B > 0$, then the two collisionless modes are characterized by $|\omega_0/\Omega_{*0}| > |B/2A|$ and $0 < |\omega_0/\Omega_{*0}| < |B/2A|$, respectively, with $\mathrm{sign}(\omega_0/\Omega_{*0}) = -\mathrm{sign}(B)$. The latter mode is destabilized by collisions ($\mathrm{Im}\ \delta\omega > 0$), with $\mathrm{Im}(\delta\omega)/\Omega_{*0} \sim \varepsilon^{1/2}\Pi_{ec}B^{-3/2}(\omega_{Dc}/b\Omega_{*0})^{1/2}$. The higher frequency mode is damped, with $\mathrm{Im}\ \delta\omega/\Omega_{*0} \sim \varepsilon^{1/2}P_e/(P_e + P_i + 1)$. If the collisionless mode is unstable, then collisions add real and imaginary increments of order ε to ω.

If ω_a^2 is not so large that χ_{aa}^0 dominates, then the collisionless mode is not localized in the solenoid; ω_0 can only be found as a root of a quartic, and the sign of $\mathrm{Im}(\delta\omega)$ is best established numerically.

The results obtained here can also be derived directly from a boundary layer analysis, such as has been done by Lane[2].

V. INTERMEDIATE COLLISONALITY REGIME

The intermediate collision frequency regime $R^{-1} << \nu/\omega << 1$ is analyzed by using the small-angle limits of the Legendre functions. From Ref. 7 we have

$$P_\alpha^\mu(\cos\theta) \simeq \beta^\mu J_{-\mu}(a)\ [1 + O(\theta^r)] \tag{48}$$

where $\beta \equiv (\alpha + 1/2) \cos\theta/2$, $a \equiv (2\alpha + 1) \sin\theta/2$, and $r = 2$ for $\alpha \sim 1$ or $\mu < 0$, but $r = 2-2\mu$ for $\alpha - 1 \sim 1$ and $\mu > 0$. The latter fact restricts applicability of (48). We shall in fact use (48) only for $\mu \cong 0$. Using the Legendre-function identity

$$Q_\nu^\mu(z) = \frac{\pi}{\mu}\frac{\exp(i\mu\pi)}{\sin\mu\pi}\left(P_\nu^\mu(z) - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} P_\nu^{-\mu}(z)\right)$$

and (48), and taking the limit $\mu \to 0$, we find

$$Q_\alpha(\cos\theta) \cong -\frac{\pi}{2}Y_0(a) + J_0(a)[\ln\beta - \psi(\nu+1)] \tag{49}$$

where ψ is the digamma function. Using the identity $L_\alpha^1(\cos\theta) = \partial L_\alpha/\partial\theta$ for $L = P,Q$, we find the remaining needed expansions,

$$P_\alpha^1(\cos\theta) \cong -\beta J_1(a)[1 + O(\theta^2)] \tag{50}$$

$$Q_\alpha^1 \cong \frac{\pi}{2}\beta Y_1(a) - \beta J_1(a)[\ln\beta - \psi(\alpha+1)] \tag{51}$$

Thus we obtain

$$\hat{h}_p^1(\theta) \cong -\beta_c\theta_{cs}J_1(a_c)/J_0(a_c) \tag{52}$$

$$\hat{h}_r^1(\theta) \cong -\beta_r\theta_{rs}\frac{\frac{\pi}{2}Y_1 - J_1\left(\ell n\ b - \psi + \frac{\pi}{2}\cot\frac{\pi\alpha}{2}\right)}{\frac{\pi}{2}Y_0 - J_0\left(\ell n\ b - \psi + \frac{\pi}{2}\cot\frac{\pi\alpha}{2}\right)}\quad, r=c,a \tag{53}$$

where $J_\mu \equiv J_\mu(a_r)$, $Y_\mu \equiv Y_\mu(a_r)$, $\psi \equiv \psi(\alpha_r+1)$, $\alpha_r \equiv \alpha(\theta_{rs}) \cong (a_r + 1/2)$ θ_{rs}, $\beta_r \equiv \beta(\theta_{rs}) \cong (b_r + 1/2)$, and we recall that $\theta_{rs} \cong R_{rm}^{-1/2}$. The intermediate and high collision frequency regimes are characterized by the inequality $\alpha_j \ll \theta_{rs}^{-1}$, in which case the Bessel functions can be approximated by their small-argument, asymptotic forms. Then

$$\hat{h}_p^1(\theta_{cs}) \cong -[(\alpha_c + 1/2)\theta_{cs}]^2/2 \tag{54}$$

$$\hat{h}_r^1(\theta_{cs}) \cong [\ell n\ \theta_{rs} + \gamma + \psi + (\pi/2)\cot(\pi\alpha_r/2)]^{-1}\quad, r=c,a \tag{55}$$

where γ is Euler's constant .5772···. Note that when $\alpha < 1$ the right hand side of (54) is zero to the accuracy of our approximation.

The intermediate-collisionality regime is obtained by taking the $\alpha \cong (i\omega/\nu)^{1/2} \gg 1$ limit of Eqs. (54)-(55), in which case

$$\hat{h}_r^1(\theta_{rs}) \cong 1/\ell n(\alpha_r\theta_{rs}\kappa)\quad, r=c,a \tag{56}$$

where $\kappa = -ie^\gamma/2$. In the opposite limit, $\alpha \cong i\omega/\nu \ll 1$, we obtain

$$\hat{h}_r^1(\theta_{rs}) \cong [\alpha_r^{-1} + \ell n\ \theta_{rs}]^{-1}$$

which reproduces the small θ_s (large R_{rm}) limit of the high-collisionality results derived in Sec. VI. Furthermore, use of the large argument limits for the Bessel functions reproduces the small-θ_s (large-R_{cm}) limits of the low-collisonality results obtained in Sec. IV.

Using Eqs. (54), (55), (28) and (19)-(21), one finds the collisonal density perturbation to be given by Eq. (36), but with the susceptibility coefficients given, to lowest order in ω_d/ω and ε^{-1} (note that ε, defined following Eq. (5.12), is now a large parameter), by

$$\chi^{\nu} = (1 - c_2\, \Omega_{*0}/\omega)c_1(\Omega_{*0}/\omega) \tag{57}$$

with $c_1 = -iP_\phi(N_a/N\Omega_{*0})\langle\nu_a \hat{h}^1{}_a \hat{h}^1{}_c/(\hat{h}^1_c + \delta\hat{h}^1_a)\rangle_\phi$, and $c_2 = -iP_\phi(N_a/N\Omega_{*0}c_1)\langle\nu_a\zeta_{ea}\hat{h}^1{}_a\hat{h}^1{}_c/(\hat{h}^1_c + \delta\hat{h}^1_a)\rangle_\phi$.

The dispersion relation is again given by (42)-(44). We note that, for $R_{ca}^{-1} \ll \nu/\omega \ll 1$ (which is henceforth assumed) we have $c_1 \cong iP_\phi(N_a/N)\langle\nu_a C/\Omega_{*0}\rangle_\phi$, with $C = -[\ln(\alpha_a\theta_{as}\kappa) + \delta\ \ln(\alpha_c\theta_{cs}\kappa)]^{-1}$ (and similarly for c_2), so that $D^\nu/D^0 \sim (\varepsilon/\ln\varepsilon)\ [P_e/(P_e + P_i + 1)]$. Thus there are two cases to consider: $D^0 > D^\nu$, which applies when ion overshoot or the finite gyroradius (FLR) term b is larger than electron overshoot and ε is not too big, and $D^\nu > D^0$, which applies when electron overshoot dominates or for very large ε.

In the former case the modes are as in the weak collisionality regime but with modified growth rates, given by using (57) instead of (37) - (38) in (44) and (45). In the strongly-MHD-stable (large-ω_a) limit considered in Sec. IV, this becomes:

$$\delta\omega = -\frac{\Omega_{*0}\,[1 - c_2\Omega_{*0}/\omega_0]\;c_1}{2A + B\Omega_{*0}/\omega_0}$$

which, for the higher (lower) frequency mode considered following Eq. (47), is unstable (stable) with $\mathrm{Im}\,\omega \sim \Omega_{*0}(\varepsilon/\ell n\,\varepsilon)[P_e/(P_e + P_i + 1)]$.

In the other case, where $D^{\nu} > D^0$, in lowest order the modes are given by the roots of $D^{\nu} = 0$,

$$D^{\nu} = c_1\frac{\Omega_{*0}}{\omega}\left(1 - c_2\frac{\Omega_{*0}}{\omega}\right)\left(1 - \frac{\langle\Omega_{*i}\rangle}{\omega} - \frac{\langle\omega^2_{MHD}\rangle}{\omega^2}\right) \tag{58}$$

where

$$N\langle\Omega_{*i}\rangle \equiv N_c\Omega_{*ic} + N_a\Omega_{*ia}\ ,$$

$$N\langle\omega^2_{MHD}\rangle \equiv -N_c\gamma_c^2 + N_a\omega_a^2\ ,$$

and more generally

$$N\langle x\rangle \equiv N_c x_c + N_a x_a\ .$$

The solutions of (58) are a high-frequency mode $\omega \gg \Omega_{*0}$, an electron drift wave $\omega = c_2\Omega_{*0}$, and two flute $[\phi_a/\phi_c = 1 + O(\varepsilon^{-1}\ell n\,\varepsilon)]$ modes, which satisfy $q_{cc} + q_{aa} = 0$, or

$$\omega = \omega_f \equiv \frac{\langle\Omega_{*i}\rangle}{2} \pm \left(\frac{\langle\Omega_{*i}\rangle^2}{4} + \langle\omega^2_{MHD}\rangle\right)^{1/2}\ . \tag{59}$$

Note that $\langle\omega^2_{MHD}\rangle$ is the pressure-weighted MHD drive. Equation (57) is the dispersion relation that describes the leading-order ion finite-Larmor-radius corrections to MHD flute modes[8]. The corrections to the lowest-order frequencies are given by perturbation theory,

$$\delta(1/\omega) = -\frac{D^0}{\partial D^\nu/\partial(1/\omega)} \tag{60}$$

The mode structure can then be determined from (46). Thus we obtain the following:

The high-frequency mode is

$$\omega \cong -\frac{\Omega_{*0}c_1}{(N_cN_a/N^2) + \sum P_j}$$

$$\sim i\ \nu_a R_{ma} r/\ell n\ r \tag{61}$$

with $r \equiv 2P_\phi/[R_{ma} + (R_{ma}N/N_a)\Sigma P_j]$ so that $\ell n\ r < 0$. The mode is thus strongly damped. The mode structure is given by

$$\frac{\phi_a}{\phi_c} \cong -\frac{N_c}{N_a}\ . \tag{62}$$

The mode thus has a zero n/B-weighted line average, and is thus typically strongly peaked in the anchor.

The electron drift wave has frequency $\omega = c_2\Omega_{*0} + \delta\omega_{ed}$ where

$$\delta\omega_{ed} = \frac{ic_2^2\Omega_{*0}^2 D^0(c_2\Omega_{*0})}{(P_\phi N_a/N)\langle\nu_a C\rangle_\phi\left(1 - \frac{\langle\Omega_{*i}\rangle}{c_2\Omega_{*0}} - \frac{\langle\omega^2_{MHD}\rangle}{c_2^2\Omega^2_{*0}}\right)} \tag{63}$$

with $D^0(\omega)$ given by Eqs. (43) and (41). For the typical case where P_e, $P_i >> N_a/N$, or for the case of a strongly MHD-stable machine where $\langle\omega^2_{MHD}\rangle >> \Omega^2_{*0}$, (63) simplifies to

$$\delta\omega_{ed} \cong \frac{ic_2^2\Omega_{*0}^2}{\langle\nu_a C\rangle_\phi P_\phi N_a/N}\left[\sum P_j\left(1 - \frac{\Omega_{*jp}}{c_2\Omega_{*0}}\right) + G\right] \tag{64}$$

$$\sim \frac{i\Omega_{*0}\ell n\ \varepsilon}{\varepsilon}\ \frac{P_e + P_i + N_c/N}{P_e}$$

where

$$G = \chi_{cc}(\omega = c_2\Omega_*) = \frac{N_c}{N}\left(1 + \frac{\langle T_{ic}\rangle}{c_2 T_e} + \frac{{\gamma_c}^2}{c_2^2\Omega_{*0}^2}\right) > 0$$

The mode structure is, from (46), $\phi_a/\phi_c \cong G/\chi_{aa}(c_2\Omega_{*0})$ where χ_{aa} is given by (41). It is seen that the mode is localized in the center cell if $\omega_a^2 >> (N_c/N_a)\gamma_c^2$, $(N_c/N_a)\Omega^2_{*ic}$.

For the flute modes (59), we can make use of the relation $\chi_{aa} = -\chi_{cc}$ in evaluating D^0 in Eq. (60). We thus find $\omega = \omega_f + \delta\omega_f$, where

$$\delta\omega_f = \frac{-\chi_{cc}^2}{\chi^\nu\ \partial(\chi_{cc} + \chi_{aa})/\partial\omega} \tag{65a}$$

$$= \frac{-iN_c^2\omega_f^2}{P_\phi N N_a \langle\nu_a C\rangle_\phi} \frac{\left(1 - \frac{\Omega_{*ic}}{\omega_f} + \frac{\gamma_c^2}{\omega_f^2}\right)^2}{\left(1 - c_2\frac{\Omega_{*0}}{\omega_f}\right)\left(2 - \frac{\langle\Omega_{*i}\rangle}{\omega_f}\right)} \tag{65b}$$

We observe that a flute mode is unstable for ω_f in the interval between $\Omega_{*i}/2$ and $c_2\Omega_{*0}$ and stable (damped) otherwise. Of the two roots of (57), one has $\omega/\langle\Omega_{*i}\rangle > 1/2$ and is always stable. The second root is unstable for small $\langle\omega_{MHD}^2\rangle$ but becomes stable for $\langle\omega_{MHD}^2\rangle > c_2^2 + c_2\langle T_i\rangle/T_e$. For example, in the limiting case where $\langle\omega_{MHD}^2\rangle \ll \Omega_{*0}^2$ and $\Omega_{*ia} = \Omega_{*ic} \equiv \Omega_{*i} = \langle\Omega_{*i}\rangle$, the two roots for ω_f are $\omega_f \cong \Omega_{*i}$ and $\omega_f \cong -\langle\omega_{MHD}^2\rangle/\Omega_{*i}$, and $\delta\omega_f$ becomes

$$\delta\omega_f \cong \frac{-i}{\langle\nu_a C\rangle_\phi} \frac{N_c^2 N_a}{N^3 P_\phi} \frac{\left(\gamma_c^2 + \omega_a^2\right)^2}{\Omega_{*i}^2} \left(1 + c_2\frac{T_e}{T_i}\right)^{-1}$$

$$\sim \frac{-i\ell_a^2 \ell n\ \varepsilon}{P_e\ \varepsilon} \frac{\left(\gamma_c^2 + \omega_a^2\right)^2}{\Omega_{*0}^3} \frac{T_e^2}{T_i^2}$$

and

$$\delta\omega_f \cong \frac{iN_a N_c^2 T_e}{P_\phi N^3 \langle\nu_a C\rangle_\phi c_2 T_i} \frac{\left(\gamma_c^2 + w_a^2\right)^2}{\Omega_{*0}^2}$$

$$\sim \frac{i\ell_a^2 \ell n\ \varepsilon}{P_e\ \varepsilon} \frac{\left(\gamma_c^2 + w_a^2\right)^2}{\Omega_{*0}^3} \frac{T_e}{T_i}$$

respectively. The former mode is stable; the latter mode is unstable.

Another simple limit is obtained when $\langle\omega^2_{MHD}\rangle >> \Omega^2_{*i}$, in which case $\omega_f \cong \pm(\langle\omega^2_{MHD}\rangle)^{1/2}$ and both modes are damped.

We note that in both the low and intermediate collisionality regimes temperature gradients affect stability quantitatively but not qualitatively, through the values of Ω_{*ia}, Ω_{*ic}, Ω_{*jp} and C_2. This is in contrast to the high-frequency regime considered in the next section, where the presence of a temperature gradient alters the order of magnitude of the growth rate for the electron drift wave.

VI. THE COLLISIONAL LIMIT, $\nu/\omega >> 1$.

In the limit of large collision frequency, $\nu/\omega >> 1$, the perturbed distribution function must satisify

$$C(h) \approx 0 . \qquad (66)$$

If the full, linearized collision operator were employed, Eq. (66) would imply that h is nearly Maxwellian at all energies. When collisions are modeled by a Lorentz operator, then Eq. (66) only implies that h is nearly independent of pitch-angle. If there are significant variations in the equilibrium potential between the center cell and the anchor cell, then the Lorentz collision operator makes a qualitative error in that the Lorentz operator will not allow the perturbed potential in the center cell to effect the distribution of electrons that haven't sufficient energy to escape from the potential peak in the anchor cell. A collision operator that includes energy diffusion and drag as well as pitch angle scattering will allow a perturbed potential in one cell to effect the distribution of particles that are energetically confined to the other cell. Hence, the Lorentz model only properly describes the collisional limit when there are no significant variations in the equilibrium potential between the center cell and the anchor cell. In this section we restrict ourselves to this case, and ignore equilibrium potential variations. The collisional limit in the presence of strong potential variations $(e\phi_{ca}/T_e >> 1)$ is considered elsewhere.[4]

We adopt the ordering ω/ν, $f_{pj} \equiv (1-\lambda_{js})$, ω_d/ω, $b \approx O(\Delta) << 1$. The index of the Legendre functions, α, is then order Δ, and we may write

$$P_\alpha^0(\lambda) \approx 1 + \alpha \ln\left(\frac{1+\lambda}{2}\right) + \cdots , \tag{67}$$

$$Q_\alpha^0(\lambda) \approx \ln\left(\frac{1+\lambda}{1-\lambda}\right) + \cdots , \tag{68}$$

$$P_\alpha^1(\lambda) \approx -\alpha(1-\lambda^2)^{1/2}\left\{\frac{1}{1+\lambda} - \alpha \, \frac{\ln[(1+\lambda)/2]}{1-\lambda} + \cdots \right\} , \tag{69}$$

$$Q_\alpha^1(\lambda) \approx -(1-\lambda)^{-1/2}\left\{1 - \alpha \left[\ln\left(\frac{1-\lambda^2}{4}\right) + \lambda \ln\left(\frac{1+\lambda}{1-\lambda}\right)\right] + \cdots \right\} ; \tag{70}$$

where Eqs. (67) and (68) may be found in Abramowitz and Stegun,[9] while Eqs. (69) and (70) are derived in the Appendix. In each region of phase space the index of the Legendre functions, α_j, is given by Eq. (10).

The collisonal perturbation in the distribution function then satifies

$$\hat{h}_p(\lambda_c) \simeq 1 + \alpha_p \ln\left(\frac{1+\lambda_c}{1+\lambda_{cs}}\right) + O(\Delta^2) \ , \tag{71}$$

and

$$\hat{h}_p^1(\lambda_{cs}) \simeq -\alpha_p(1-\lambda_{cs}) + O(\Delta^3) \tag{72}$$

in the passing region of phase space; while

$$\hat{h}_j(\lambda_j) \simeq 1 + \frac{\alpha_j}{2} \ln\left(\frac{1-\lambda_j^2}{1-\lambda_{js}^2}\right) + O(\Delta^2) \ , \tag{73}$$

and

$$\hat{h}_j^1(\lambda_{js}) \simeq \alpha_j \lambda_{js}\left[1-\frac{\alpha}{2}\ln\left(\frac{f_{pj}}{2}\right)\right] + O(\Delta^3) \tag{74}$$

in the trapped regions of phase space. The subscript j takes on the values c and a for center and anchor cell trapped particles respectively, while

$$\alpha_j \simeq i\frac{\omega}{\nu}\left(1 - \frac{\omega_{Dj}}{\omega} - i\frac{\omega}{\nu}\right) \ . \tag{75}$$

When α appears without a subscript the dependence on the drift frequency may be ignored, yielding $\alpha \simeq i\omega/\nu$. Note that the equilibrium plasma density in the center cell and the anchor cell must be equal if the electron distribution is Maxwellian and there are no equilibrium potential variations. Hence, $\nu_a = \nu_c$.

Inserting Eqs. (71) thru (74) into (19) we find, after a great deal of algebra, that

$$A_p = -h_p^0 - \left(1-\frac{\Omega_{*0}}{\omega}\right)\left[\left(1+\frac{\langle\omega_D\rangle}{\omega}\right)\langle\phi\rangle - \frac{\alpha}{2}\frac{\delta}{(1+\delta)^2}\ln\sigma\ (\phi_c - \phi_a)\ \right] . \quad (76)$$

It is now straight-foward, though tedious, to calculate the perturbed electron density in each cell, and hence the susceptibility tensor describing the collisional electron responce. It is convenient to sum the collisional susceptibility of the electrons together with the leading order (in our small parameters f_p, ω_D/ω, and b) collisionless susceptibilities of both electrons and ions to obtain the leading order susceptibility,

$$\overleftrightarrow{\chi}^{\nu} \equiv \chi^{\nu} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \quad (77)$$

χ^{ν} is now given by

$$\chi^{\nu} = \left[\left(1-\frac{\Omega_{*0}}{\omega}\right)\left(1+i\frac{\omega}{\nu_*}\right)-i\frac{3}{2}\frac{\Omega_{*T}}{\nu_*}\right]\frac{N_a N_c}{bN^2} , \quad (78)$$

here

$$\nu_* = \frac{(2\pi)^{1/2}}{16}\left(\frac{v^2}{T}\right)^{3/2}\nu_e\left[\frac{1}{2}\ln\left(\frac{2}{f_{pa}}\right)-\frac{1}{2}\frac{N_a}{N}\ln\sigma-1\right]^{-1} ,$$

and $\Omega_{*T} \equiv \eta\Omega_{*0}$. The remaining piece of $\overleftrightarrow{\chi}$ has the same structure as $\overleftrightarrow{\chi}^{(0)}$ of Eq. (41), where χ_{cc}, χ_{ca}, and χ_{aa} are given by

$$\chi_{cc}^{(0)} = \frac{N_a N_c}{N^2}\tau\left[\frac{\omega_{MHD,a}^{(i)2} - \omega_{MHD,c}^{(i)2}}{\omega^2} + \frac{\tau_a^2\omega_{MHD,a}^{(i)2} - \tau^2\omega_{MHD,c}^{(i)2}}{\tau^2\Omega_{*i}\omega} + \frac{\Omega_{*0}}{\omega}\left(\frac{1}{\tau} - \frac{1}{\tau_a}\right)\right]$$

$$+ \frac{\tau}{\omega^2}\frac{N_c}{N}\left(\omega^2 - \omega\langle\Omega_{*i}\rangle - \langle\Omega_{MHD}^2\rangle\right) , \qquad (79a)$$

$$\chi_{ca}^{(0)} = \left(1 - \frac{\Omega_{*0}}{\omega} - \frac{\Omega_{*T}}{\omega}\right)\frac{\langle\omega_D\rangle}{b\omega}\frac{N_a N_c}{N^2} + P_i\left(1 - \frac{\Omega_{*i}}{\omega}\right) , \qquad (79b)$$

$$\chi_{aa}^{(0)} = -\frac{N_a N_c}{N^2}\tau\left[\frac{\omega_{MHD,a}^{(i)2} - \omega_{MHD,c}^{(i)2}}{\omega^2} + \frac{\tau_a^2\omega_{MHD,a}^{(i)2} - \tau^2\omega_{MHD,c}^{(i)2}}{\tau^2\Omega_{*i}\omega} + \frac{\Omega_{*0}}{\omega}\left(\frac{1}{\tau} - \frac{1}{\tau_a}\right)\right]$$

$$+ \frac{\tau}{\omega^2}\frac{N_c}{N}\left(\omega^2 - \omega\langle\Omega_{*i}\rangle - \langle\Omega_{MHD}^2\rangle\right) , \qquad (79c)$$

where ion quantities are indicated by the superscript (i), and we have allowed for different ion temperatures in the center and anchor cells.

The structure of $\overleftrightarrow{\chi}^\nu$ in this section is the same as that of $\overleftrightarrow{\chi}^\nu$ in Secs. IV and V. The determinant of $\overleftrightarrow{\chi}^\nu$ again vanishes. Hence, to leading order the dispersion relation is given by

$$\left(\omega^2 - \langle\Omega_{*i}\rangle\omega - \langle\omega_{MHD}^2\rangle\right)\left[\left(1-\frac{\Omega_{*0}}{\omega}\right)\left(1+i\frac{\omega}{\nu_*}\right) - i\frac{3}{2}\frac{\Omega_{*T}}{\nu_*}\right]$$

$$= -\frac{b\omega^2}{\tau}\frac{N_a N_c}{N^2}\left(\chi_{cc}\chi_{aa} + \chi_{ca}(\chi_{cc} + \chi_{aa})\right) . \qquad (80)$$

The right hand side of Eq. (80) is small. Hence, at leading order the dispersion relation may be obtained by setting the left hand side of Eq. (80) to zero. We find three modes. A mode analagous to the dissipative trapped electron mode in toriodal systems,[10] which satifies the dispersion relation

$$\omega \simeq \Omega_{*0} + i\frac{3}{2}\frac{\Omega_{*0}\Omega_{*T}}{\nu_*}, \tag{81}$$

together with a pair of modes with frequencies given by Eq. (59). These three modes are obvious extensions of three of the modes identified in the $D^{\nu} > D^0$ limit of the intermediate-collisionality regime (Section V). The fourth root identified in Sec. V, the high-frequency damped mode, always satisfies $\nu/\omega < 1$ and so doesn't appear in the high-collisionality analysis.

It is interesting to note that the growth rate of the electron drift wave identified in the intermediate-collisionality regime (Eq. 64) differs from that of the corresponding high-collisionality mode (Eq. 81) in that the growth rate for the latter vanishes at leading order if the tempereature gradient vanishes.

As in Sec. V, the lowest order dispersion relation of the two flute-like modes [i.e., the modes satisfying Eq. (59)] may be written symbolically as $\chi_{aa} = -\chi_{cc}$. Using this relation in the right hand side of Eq. (80), the growth rate for these modes may then be written in the form

$$\gamma = -\frac{b}{\tau}\frac{N_aN_c}{N^2}\frac{\omega^2}{\nu_*}\frac{\chi_{cc}^2}{(2-\langle\Omega_{*i}\rangle/\omega)}\left(\frac{\omega}{\omega-\Omega_{*0}}\right)^2\left[1-\frac{\Omega_{*0}}{\omega}(1+\frac{3}{2}\eta)\right] . \tag{82}$$

It is evident from Eq. (82) that the growth rate can be positive only when the real part of the frequency falls in the range

$$\frac{1}{2}\frac{\langle\Omega_{*i}\rangle}{\Omega_{*0}} < \frac{\omega}{\Omega_{*0}} < 1 + \frac{3}{2}\eta , \tag{83}$$

a result similar to that derived in Sec. V. This condition is never satisfied when the sign in front of the square root in Eq. (59) is chosen to be the same as the sign of $\langle\Omega_{*i}\rangle$. The remaining flute mode will be unstable when $\langle\omega^2_{MHD}\rangle$ lies in the range

$$0 < \langle\omega^2_{MHD}\rangle < \Omega^2_{*0}\left(1+\frac{3}{2}\eta\right)\left[1+\frac{3}{2}\eta - \frac{\langle\Omega_{*i}\rangle}{\Omega_{*0}}\right] . \tag{84}$$

In the limit $\langle\omega^2_{MHD}\rangle \ll \langle\Omega_{*i}\rangle^2$ we find that the mode with $\omega_f \simeq \langle\Omega_{*i}\rangle$ is damped. The damping rate is given by

$$\gamma \simeq -b\frac{N_aN_c}{N^2}\frac{\omega^{(i)^4}_{MHD,a}}{\Omega_{*i}\nu_*}\left(\frac{\tau^2}{1+\tau}\right)^2\left(1+\tau+\frac{3}{2}\eta\right) \tag{85}$$

when $\omega^{(i)^2}_{MHD,a} \gg \Omega^2_{*i}$; and by

$$\gamma \simeq -b\frac{N_aN_c}{N^2}\frac{\langle\Omega_{*i}\rangle^2}{\nu_*}\frac{1+\tau+3/2\,\eta}{\tau^2(1+\tau)^2} \tag{86}$$

when $\Omega_{*i}^2 \gg \omega_{MHD,a}^{(i)2}$, $\omega_{MHD,c}^{(i)2}$.

The remaining flute mode has a frequency $\omega_f \simeq -\langle\omega_{MHD}^2\rangle/\langle\Omega_{*i}\rangle$, and a growth rate

$$\gamma \simeq b \frac{N_a N_c}{N^2} \frac{\langle\Omega_{*i}\rangle^2}{\nu_*} \frac{\omega_{MHD,a}^{(i)2}}{\langle\omega_{MHD}^2\rangle} . \tag{87}$$

In the opposite limit, $\langle\omega_{MHD}^2\rangle \gg \Omega_{*i}^2$, the real frequency of the two flute modes is given by $\omega_f \simeq \pm \langle\omega_{MHD}^2\rangle^{1/2}$. Both of these modes are damped. The damping rate is

$$\gamma \simeq -b \frac{N_a N_c}{N^2} \frac{\omega_{MHD,a}^{(i)4}}{\nu_* \Omega_{*i,a}^2} \frac{\tau_a^2}{\tau} . \tag{88}$$

VII. CONCLUSION

The effects of electron pitch-angle scattering on trapped particle modes in tandem mirrors have been considered for arbitrary ν/ω. When $\varepsilon \equiv \nu R/\omega$ is small, electron collisional dissipation yields small, destabilizing corrections to the collisionless trapped particle modes considered previously. At larger values of ε the collisions substantially alter the dispersion relation. The modes found then include an electron drift wave, two flute modes and a high-frequency, nearly-purely damped mode. The most dangereous mode is then the electron drift wave, with a growth rate $\gamma \sim \Omega_{*0}{}^2/\nu$. It is likely that current tandem mirror experiments operate in this collisional regime, $\varepsilon \gg 1$. These experiments should observe an unstable spectrum propagating near the electron diamagentic drift velocity relative to the $\underset{\sim}{E} \times \underset{\sim}{B}$ electric-drift velocity of the plasma. The growth rate of these modes increases with the square of the azimuthal mode number. Hence, we may expect short scale length turbulence much like that observed in tokamaks. This turbulence may lead to enhanced radial transport. [This is in contrast to the collisionless, or weakly collisional theory, which predicts that the most unstable modes will propagate in the <u>ion</u> diamagnetic drift direction (relative to the $\underset{\sim}{E} \times \underset{\sim}{B}$ drift) with a growth rate that does not dramatically increase with the azimuthal mode number.] Broadband turbulence with frequencies of the order of ω_* have been observed on TMX-U.

In this paper, ion collisionality has been neglected as the ion collision frequency is so much smaller than that of the electrons. The ion collisional response can be straightforwardly derived following the procedures used here for electrons. Ion collisionality effects do not become important until the electrons are strongly collisonal; results for weakly collisional ions and strongly collisional electrons are given by Lane[2].

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APPENDIX: SMALL INDEX EXPANSIONS OF LEGENDRE FUNCTIONS.

We evaluate Legendre functions at small index, υ, by making a Taylor series expansion of the Legendre functions, P_υ and Q_υ, about $\upsilon=0$. Hence, we need expressions for

$$\frac{d}{d\upsilon} P_\upsilon^1(\lambda)\Big|_{\upsilon=0} , \quad \frac{d}{d\upsilon} Q_\upsilon^1(\lambda)\Big|_{\upsilon=0} , \text{ and } \frac{d^2}{d\upsilon^2} P_\upsilon^1(\lambda)\Big|_{\upsilon=0} .$$

The Legendre functions satisify the equation

$$\frac{d}{d\lambda} (1-\lambda^2) \frac{d}{d\lambda} L + \upsilon(\upsilon+1) L = 0 , \tag{A1}$$

where $L = P_\upsilon^0, Q_\upsilon^0$. Taking the υ-derivative of this equation, and evaluating it at $\upsilon=0$, we find

$$\frac{d}{d\lambda} (1-\lambda^2) \frac{d}{d\lambda} L_\upsilon = -L , \tag{A2}$$

and

$$\frac{d}{d\lambda} (1-\lambda^2) \frac{d}{d\lambda} L_{\upsilon\upsilon} = -2L - 2L_\upsilon ; \tag{A3}$$

where the subscript on L indicates the derivative with respect to υ.

Equations (A1) and (A2) may be integrated over λ to obtain L_υ and $L_{\upsilon\upsilon}$. A boundary condition at $\lambda=0$ may be obtained from Abramowitz and Stegun[9], Eqs. (8.6.3) and (8.6.4). We find

$$\frac{d}{d\upsilon} P_\upsilon^0(0)\Big|_{\upsilon=0} = -\ln 2 \ , \quad \frac{d}{d\upsilon} P_\upsilon^1(0)\Big|_{\upsilon=0} = -1 \ , \quad \frac{d}{d\upsilon} Q_\upsilon^0(0)\Big|_{\upsilon=0} = -\frac{\pi^2}{4} \ ,$$

$$\frac{d}{d\upsilon} Q_\upsilon^1(0)\Big|_{\upsilon=0} = -\ln 2 \ , \text{ and } \frac{d^2}{d\upsilon^2} P_\upsilon^1(0)\Big|_{\upsilon=0} = -2\ln 2 \ .$$

Integrating Eq. (A2) once, and using the relations

$$P_\upsilon^1(\lambda) = -(1-\lambda^2)^{1/2} \frac{d}{d\lambda} P_\upsilon^0(\lambda) \quad \text{and} \quad Q_\upsilon^1(\lambda) = -(1-\lambda^2)^{1/2} \frac{d}{d\lambda} Q_\upsilon^0(\lambda) \ ,$$

we find

$$\frac{d}{d\upsilon} P_\upsilon^1(\lambda)\Big|_{\upsilon=0} = -\left(\frac{1-\lambda}{1+\lambda}\right)^{1/2} \ , \tag{A4}$$

and

$$\frac{d}{d\upsilon} Q_\upsilon^1(\lambda)\Big|_{\upsilon=0} = (1-\lambda^2)^{-1/2}\left[\ln\left(\frac{1-\lambda^2}{4}\right) + \lambda\ln\left(\frac{1+\lambda}{1-\lambda}\right)\right] \ . \tag{A5}$$

A further λ integration then yields the result

$$\frac{d}{d\upsilon} P_\upsilon^0(\lambda)\Big|_{\upsilon=0} = \ln\left(\frac{1+\lambda}{2}\right) \ , \tag{A6}$$

which may also be found in Abramowitz and Stegun [Eq. (8.6.20)]. Finally, we integrate Eq. (A3) once to obtain

$$\frac{d^2}{d\upsilon^2} P^1_\upsilon(\lambda)\Big|_{\upsilon=0} = 2\left(\frac{1+\lambda}{1-\lambda}\right)^{1/2} \ln\left(\frac{1+\lambda}{2}\right) , \tag{A7}$$

which is the final result required.

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